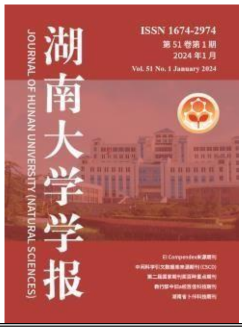

湖南大学学报（自然科学版）
Journal of Hunan University (Natural Sciences）

第 53 卷 第 1 期
2 0 2 5 年

Available online at http://jonuns.org

Vol. 53 No. 1
2025

Open Access Article

10.5281/zenodo.14921151

The Problem of Managing Reentrangeable Resources on Network Graphs with Homogeneous Intensity Functions

O.A. Kosorukov*

Lomonosov Moscow State University, Russian Presidential Academy of National Economy and Public Administration, Moscow, Russia

E-mail: kosorukovoa@mail.ru

D.V. Lemtyuzhnikova

Trapeznikov Institute of Control Sciences of the Russian Academy of Sciences;

MAI (National Research University), Moscow

E-mail: darabbt@gmail.com

D.A. Maksimov

Plekhanov Russian University of Economics, Moscow, Russia

E-mail: maksimov.da@rea.ru

Abstract: The article considers the problem of optimal distribution of resources allocated for the execution of a certain set of interrelated tasks, according to the criterion of minimizing the execution time of all tasks. In contrast to the traditional consideration of a separable resource, a reentrant resource is considered, i.e. one that can be reused. A formalization of this problem is given in a dynamic statement. For the case of homogeneous linear productivity functions, an analytical solution is given and justified. A demonstration of the non-optimality of some heuristic algorithms is given. A geometric optimality criterion is given for the case of two independent jobs. An example of complete internal resource switching is constructed on its basis.

Keywords: optimization, resource allocation, network schedule, execution intensity functions.

*Corresponding author: kosorukovoa@mail.ru

The research results were obtained through the support of the Russian Science Foundation, project no. 22-71-10131.

Introduction

A significant amount of literature and scientific publications are devoted to network planning problems. In particular, the formulations of optimization problems on network graphs are given in the monograph [1], however, unlike the dynamic problems of resource allocation considered in this article, they are static. The problems studied in this paper differ in their formulation from the problems given in [1], which considers separable or, as they are often called, material and technical resources. This article considers another type of resource, namely, resources that allow repeated use in different jobs at different points in time, for example, personnel or equipment. This type of resource will be called reentrant; in the literature it is often called non-storable or a "capacity" type resource. Let us give a brief overview of the results for problems of this type. The book [2] states that there are no exact algorithms for finding the optimal resource allocation in the general case, but there are a number of heuristic algorithms. The general case is understood as a set of works specified by an arbitrary network schedule, the presence of several resources of known volumes, each work is performed by one type of resource, apparently the operations are performed with a fixed intensity (i.e., the intensity does not change from start to finish). Results are given for independent works with concave intensity functions. It is noted that in the case when one work is performed by several types of resources on one, but with a fixed distribution of shares of resource use (each work has its own structure), results 1 and 2 remain valid. It is asserted that if the intensity functions are not concave (the general case), then there always exists an optimal solution containing no more than n intervals of constancy, i.e., the intensities do not change on them. The result is given for independent works with convex intensity functions. For the case of fixed intensities and independent works, an optimization algorithm is considered, an analogue of the simplex method for a specific class of problems.

The article [3, 4] considers the case of projects with dependent works (an arbitrary network schedule), which are performed with fixed a priori specified intensities. The average weighted time of completion of all project work is considered as a criterion for the schedule efficiency.

The work [5] presents mathematical models that allow optimization of the network schedule when distributing material or time resources. The methods for determining the dependencies between the time of execution of network schedule works and the volume of resources spent on this work are considered, linear and hyperbolic dependencies are considered. Further, mathematical models are presented that allow optimization of the network schedule in time so that the resources allocated to the project do not exceed the standard indicators. A mathematical model for minimizing the resources allocated to the project, in conditions of observing time limits, is also presented.

In article [6], the problem of reentrant resource allocation during execution of a set of interdependent tasks presented in the form of a network schedule is considered. A linear dependence of the time of task execution on the resources used is assumed. An algorithm for constructing a solution for tasks with a predetermined sequence of events in the network schedule of a set of tasks is substantiated. An algorithm for reducing a general problem to an auxiliary problem with ordered times of events is proposed, as well as an algorithm for constructing an optimal solution to the original problem. The convergence of this algorithm is due to the finiteness of iterations at each stage. The overall computational complexity of the algorithm can be estimated as O(n2), where n is the number of nodes in the original network schedule.

The article [7] considers the issues of estimating the parameters of the network project management models. The features of constructing network models under various conditions of network planning are described. The influence of model aggregation on the accuracy of calendar plans is estimated.

The work [8] is devoted to similar in formulation problems of dynamic resource management. It considered a hydrostatic model. Based on physical analogies and laws, it was shown that the optimal use of resources is achieved with a minimum of the functional, which expresses the value of potential energy.

The paper [9] considers the problem of

distributing teams of specialists who jointly perform certain tasks (consulting services, assessment of conformity or non-conformity of products, goods or services to the necessary requirements, etc.). The team performing the work consists of many specialists of different types (specialties), the number of which is limited. The problem is to develop a work plan for the teams, in which the time for completing all the work is minimal. The problem is reduced to a linear programming problem. Heuristic algorithms and special cases of the problem are also considered.

The work [10] considered the problems of scheduling theory with non-storable resources and algorithms for their solution based on branch and bound methods.

We also note several works by foreign authors. The article [11] considers the problems of scheduling projects with limited resource and material flows (Resource-Constrained Multi-project Scheduling Problem, RMCPSP). The publication [12] provides a comparison of some resource allocation algorithms on network graphs. The work [13] presents an overview of the results on issues of dynamic resource allocation on network graphs, and also provides a detailed bibliography.

1. Statement of the problem with separable resources

Let us present the statement of the classical problem of optimal resource allocation on a network schedule in the deterministic case [1]. Let a network schedule with events $z_1, \dots, z_n$ and arcs-jobs $l_1, \dots, l_m$, be given, where z_1 is the beginning of all jobs, z_n is their end. Let us also assume that the execution times of all jobs are functions of resource allocations. In this case, we will assume that a set X of resource distributions is given, and as soon as a resource distribution x∈X is selected, the functions $\varphi_j(x)$ are immediately determined, i.e., the execution times of jobs l_j, j=1,..., m. We will assume that the functions $\varphi_j(x)$, j=1,..., m, are continuous non-negative functions, and the set X is a compact set of the Euclidean space $E^{(k)}$.

Let us formulate the problem of finding such a resource distribution that the execution time of the entire set of works is minimal for a given resource distribution. If the resource distribution x∈X is chosen, then the minimum execution time of the entire set of works will be determined from the solution of the following problem (1).

$$\min_t \left(t_n - t_1\right), \tag{1}$$

$$t_{n_2(j)} - t_{n_1(j)} \geq \varphi_j(x),\ t = (t_1, \dots, t_n),\ j = 1, \dots, m,$$

where $n_1(j)$ is the number of the vertex that is the beginning of the arc j, and $n_2(j)$ is the number of the vertex that is the end of the arc j.

If we want to choose such a resource distribution that the execution time of works is minimal for all resource distributions, then we need to solve the problem $\min_x \min_t \left(t_n - t_1\right)$ under the constraints of problem (1). Combining two consecutive minima into one, we finally obtain

$$\min_{x,t} \left(t_n - t_1\right) \tag{2}$$

$$t_{n_2(j)} - t_{n_1(j)} \geq \varphi_j(x),\ t = (t_1, \dots, t_n),\ j = 1, \dots, m,\ x \in X.$$

We especially note the case when one type of resource is distributed, for example, money. In this case, problem (2) will have the form

$$\min_{x,t} \left(t_n - t_1\right), \tag{3}$$

$$t_{n_2(j)} - t_{n_1(j)} \geq \varphi_j\left(x\right),\quad t = \left(t_1, \dots, t_n\right),$$

$$x = (x_1, \dots, x_m),\ x_j \geq 0;\quad \sum_{j=1}^{m} x_j \leq A,\ j = 1, \dots, m,$$

where A is a predetermined amount of continuous resource that is distributed among all jobs. This type of resource does not imply reuse, that is, being distributed to one of the jobs, it cannot be used to another. We will call this type of resource separable. Problem (3) is a separable problem, generally speaking, of nonlinear mathematical programming, for which numerous solution methods are known depending on the properties of the functions $\varphi_j(x)$ [1]. Next, we will consider a similar problem, but with a different type of resource.

2. Statement of the problem with reentrant resources

This article considers another type of resource, namely, resources that allow reuse in different jobs at different points in time, for example, people or equipment. We will call this type of resource reentrant. Let us formalize the problem of a similar problem (3), but for a reentrant resource.

Similar to problem (3), we will consider a single resource in the amount of A, assuming it is infinitely divisible.

$$\min_{t,\ u} t_n \quad (4)$$

$$\int_{t_{n_1(j)}}^{t_{n_2(j)}} W_j(u_j(z))dz \geq A_j\,, \quad j = 1, \dots, m,$$

$$u_1(t) + \dots + u_m(t) \leq A, t \in [0, T],$$

$$t_j \geq 0, j = 1, \dots, m.$$

where T is a sufficiently large number, obviously greater than the execution time of the set of works, A_j is the volume of the j-th work, $u_j(t)$ is the amount of resource allocated to execute the j-th work at time t, $W_j(u_j(t))$ is the intensity of execution of the j-th work at time t depending on the resource involved at time t to execute this work. We assume that the functions W() are continuous, positive, monotonically increasing functions on the set [0; A]. Since their argument - the volume of resource allocated to the corresponding work, depends on time, we obtain a complex function of time. The meaning of the functions W() under consideration assumes that their integration over any time interval allows us to determine the volume of the corresponding work performed during a given time interval in units of calculation of the corresponding work.

As a class of functions $u_j(t)$, it is advisable to consider non-negative piecewise continuous functions on the interval $t \in [0,T]$, if we admit the possibility of instantaneous redistribution of the resource. The functions of instantaneous intensities $W_j()$, i.e. the volume of work performed per unit of time, in this formulation of the problem depend only on the instantaneous volume of allocated resources and do not depend on the stage of work execution. Such a simplification is adequate only in the case of homogeneous works.

Formalization (4) is an optimization problem of complex design, since some of the optimization variables form a vector in a finite-dimensional space $(t_1, \dots, t_n) \in R^n$, and some form a vector function from an infinite-dimensional space $(u_1(t), \dots, u_m(t)) \in U$. Such problems do not fit into any classes of optimization problems supported by analytical or numerical methods. In [14], this problem was formalized by the author as an optimal control problem, namely, the classical problem of speed.

3. Problem with homogeneous linear intensity functions

Let us consider problem (4) with the intensity functions of the following form $W_j(u_j(t)) = a_j u_j(t)$. In [14], the authors considered a problem of this type for two independent jobs as an example. Using the formalization of this problem in the form of an optimal control problem and Pontryagin's maximum principle, an optimal solution was obtained, which was expressed functionally as $T^* = \frac{1}{A}\left(\frac{A_1}{a_1} + \frac{A_2}{a_2}\right)$. In essence, this means that in this problem there is no need to distribute resources among jobs, nor, even more so, any dynamic redistribution of resources during the execution of jobs. It is sufficient to consistently direct all available resources to each of the two jobs. Let us prove this result for an arbitrary network schedule with homogeneous linear intensity functions.

Theorem 1. For problem (4) in the case of linear homogeneous functions of the intensities of work execution $W_j(u_j(t)) = a_j u_j(t)$, j=1,…,m, the optimal time for completing a set of works is

$$T^* = \frac{1}{A}\left(\frac{A_1}{a_1} + \dots + \frac{A_m}{a_m}\right).$$

Proof.

For the case under consideration, problem (4) takes the following form (5).

$$\min_{t,\ u} t_n \quad (5)$$

$$a_j \int_{t_{n_1(j)}}^{t_{n_2(j)}} u_j(z)dz \geq A_j\,, \quad j = 1, \dots, m, \quad (6)$$

$$u_1(t) + \dots + u_m(t) \leq A, t \in [0, T], \quad (7)$$

$$t_j \geq 0, j = 1, \dots, m.$$

Constraints (6) can be replaced with strict equalities, since the resource $u_j(t)$ is a resource allocated strictly to work j, with a volume of V_j, and there is no reason to allocate it in a larger volume. Similarly, we can consider strict equality in constraint (7). Since underutilization of the resource, due to the monotonic non-decrease of the intensity functions $W_j()$, cannot speed up the project execution time. Taking this into account, we write down an equivalent form of the problem.

$$\min_{t,\ u} t_n \quad (8)$$

$$\int_{t_{n_1(j)}}^{t_{n_2(j)}} u_j(z)dz = \frac{A_j}{a_j},\ \ j = 1,\dots,m, \quad (9)$$

$$u_1(t)+\dots+u_m(t) = A, \quad t \in [0,T],$$

$$t_j \geq 0, j = 1,\dots,m.$$

The solution of problem (8), as well as problem (5), always exists by virtue of the well-known theorem on the existence of a minimum of a continuous function on a closed bounded set. Obviously, it can be stated that there exists an optimal solution to problem (8) such that

$$u_j(t) = 0, \qquad \forall\, t \notin \left[t_{n_1(j)}, t_{n_2(j)}\right]. \quad (10)$$

Next, we consider a solution (t, u) of this type. We further arrange the sequence of values of the vector t, namely, $t_{k_1}, \dots, t_{k_n}$. Obviously, $t_{k_1} = t_1$, and $t_{k_n} = t_n$. Further, from the constraints of problem (8) it follows that

$$\sum_{j=1}^{m} \int_{t_{k_{i-1}}}^{t_{k_i}} u_j(t)dt = \int_{t_{k_{i-1}}}^{t_{k_i}} (\sum_{j=1}^{m} u_j(t))dt = A(t_{k_i} - t_{k_{i-1}}), \quad i = 2, \dots, n.$$

And then, taking into account the constraints (9) and condition (10), we have

$$\sum_{i=2}^{n}\sum_{j=1}^{m} \int_{t_{k_{i-1}}}^{t_{k_i}} u_j(t)dt = \sum_{i=2}^{n} A\left(t_{k_i} - t_{k_{i-1}}\right) = A(t_n - t_1) =$$

$$= \sum_{j=1}^{m}\sum_{i=2}^{n} \int_{t_{k_{i-1}}}^{t_{k_i}} u_j(t)dt = \sum_{j=1}^{m} \int_{t_{n_1(j)}}^{t_{n_2(j)}} u_j(t)dt = \sum_{j=1}^{m} \frac{A_j}{a_j}.$$

And therefore, in the optimal solution under consideration

$$A(t_n - t_1) = \sum_{j=1}^{m} \frac{A_j}{a_j}.$$

Whence follows the statement of the theorem

$$T^* = (t_n - t_1) = \frac{1}{A}\left(\frac{A_1}{a_1} + \dots + \frac{A_m}{a_m}\right).$$

Since $\frac{1}{A}\frac{A_j}{a_j}$ is the time it takes for the full resource A to complete the task j, the time it takes to complete the task set obtained in the theorem corresponds to the sequential execution of the tasks using the full resource. The permissible order of task execution is determined by the network schedule and, generally speaking, is not unambiguous, which does not affect the time it takes to complete the task set.

4. Problem with homogeneous nonlinear intensity functions

We will begin the study by considering the simplest network graph, namely, let there be only two independent jobs with nonlinear intensity functions W_1 (u) and W_2 (u), with volumes A_1 and A_2, respectively. For the study, we will use an optimization model in the Excel environment. Since we cannot perform infinite-dimensional optimization in Excel for the entire class of functions, we will consider class of functions u(t) are piecewise constant functions with at most one switching. In this regard, we introduce some additional notation and formalize the problem in this class of functions.

$$\min_{u_1,u_2,t,T} T \quad (11)$$

$$W_1(u_1)t + W_1(u_2)(T-t) \geq A_1,$$
$$W_2(A-u_1)t + W_2(A-u_2)(T-t) \geq A_2,$$
$$0 \leq t \leq T,$$
$$0 \leq u_1 \leq A,$$
$$0 \leq u_2 \leq A.$$

Where t is the switching moment, at t=0 or t=T there is no switching, u_1 is the resource for performing job 1 in the time period from 0 to t, u_2 is the resource for performing job 1 in the time period from t to T.

Next, we will test various possible algorithms for solving problem (11). To do this, we will conduct a series of computational experiments with various intensity functions and various numerical parameters and draw conclusions. Let's consider the following algorithms:

1. "Sequential execution algorithm" (SEA). This algorithm implies sequential execution of jobs with the corresponding allocation of the full volume of resources for each job. It was this algorithm that turned out to be optimal for the linear case considered by us above.
2. "Synchronous completion algorithm" (SCA). It assumes that parallel independent jobs that complete at one node of the network graph are completed simultaneously. Why complete any of these jobs before the others? After all, this

will not cause the final event to occur earlier. It is possible to reduce a part of the resource allocated for the execution of such "early" work and give it to speeding up longer works.

3. "Maximum Intensity Algorithm" (MIA). It assumes that at each moment of time all works that can be executed are considered and the resource is distributed between them in such a way that the total intensity will be maximum. We expect that maximizing the total productivity at each moment of time will lead us to the optimal solution.

4.1. Testing the APV algorithm

Example 1.1

$$W_1(u) = u^2, \quad W_2(u) = 2u^2,$$
$$A = 6,\ A_1 = 20, \quad A_2 = 40.$$

As can be seen from the analysis of the solution results, first the full resource performed job 2 until the time 0.56, then there was a full switch to job 1, which was completed at the time 1.11. Thus, the APV algorithm was implemented.

Conclusion for example 1.1: The APV algorithm can lead to an optimal solution.

Example 1.2

$$W_1(u) = \sqrt{u}, \quad W_2(u) = 2\sqrt{u},$$
$$A = 6,\ A_1 = 20, \quad A_2 = 40.$$

As can be seen from the solution results, there is no resource switching, the jobs are performed in parallel with the use of resources 3 and 3 and are completed simultaneously at the time 11.55. Thus, the ASZ algorithm is implemented. Let us calculate the result of the APV algorithm for example 1.2: $\frac{20}{\sqrt{6}} + \frac{40}{2\sqrt{6}} = 16{,}33$. As we can see, it is not optimal.

Conclusion for example 1.2: The APV algorithm may not lead to an optimal solution.

General conclusion for the APV algorithm: The APV algorithm may or may not lead to an optimal solution.

4.2. Testing the ASZ algorithm

Based on example 1.2, we can conclude: The ASZ algorithm may lead to an optimal solution. Let us calculate the result of applying the ASZ algorithm for example 1.1: $\frac{20}{u^2} = \frac{40}{2(6-u)^2},\ u = 3$. Hence, the project execution time is 20/9=2.22. According to example 1.1, this time is not optimal, and therefore, the ASZ algorithm does not lead to an optimal solution.

General conclusion for the ASZ algorithm: The ASZ algorithm may or may not lead to an optimal solution.

4.3. Testing the AMI algorithm

Example 3.1

$$W_1(u) = 20\sqrt{u}, \quad W_2(u) = u^2,$$
$$A = 6,\ A_1 = 20, \quad A_2 = 40.$$

Now we will solve the example using the AMI algorithm. We will find the resource allocation that provides the maximum integral intensity of work. The integral intensity function is *F(u)*=$20\sqrt{u} + (6-u)^2$, then we will calculate the derivative of this function as $F'(u) = \frac{20}{2\sqrt{u}} - 2(6-u)$. The stationary point, which corresponds to the maximum of the function, is the point u=1. This means that according to the AMI algorithm, it is necessary to start performing work 1 with resource 1, and work 2 with resource 5. According to the AMI algorithm, this resource allocation is maintained until the first of the works is completed. The completion time of work 1 is 1, and that of work 2 is 1.6. This means that the redistribution resources will occur in time 1, after the completion of work 1. By this time, the remainder of work 2 will be 15, which will begin to be performed by the full resource 6 and, accordingly, will be completed in time 0.42. Thus, the total project time will be 1.42, which is more than the optimal time of 1.35.

Conclusion for example 3.1: The AMI algorithm may not lead to an optimal solution.

It can be noted that if the volume of work 2 were not 40, but 25, then the result of applying the AMI would not lead to an optimal solution. However, in this case, this solution would also be obtained using the ASZ algorithm.

General conclusion for the AMI algorithm: The AMI algorithm may or may not lead to an optimal solution. Thus, none of the three "reasonable" algorithms considered by us in this section is an algorithmic solution for problem (11), and therefore for the wider class of problems described in section 2.

5. The Question of the Existence of Optimal Internal Switching

We consider the same problem as in section 4, namely, let there be only two independent jobs with nonlinear intensity functions W_1 (u) and W_2 (u), with volumes A_1 and A_2, respectively, problem (11). In addition to the previously made assumptions regarding the intensity functions W(u), namely, we assume that the functions W() are continuous, positive, monotonically increasing functions on the set [0; A], we will additionally assume their differentiability.

By internal switching we mean the solution of problem (11), namely, piecewise constant functions with at most one switching, for which the constraints of problem (11) are satisfied as strict equalities: $0<t<T$, $0<u_1<A$, $0<u_2<A$ and $u_1 \neq u_2$. Substantially, this means that the redistribution of resources occurs not "in connection with the beginning" of some work and not "in connection with the end" of some work, but in the process of performing both works. For definiteness, we assume that $u_1<u_2$. This assumption does not violate the generality of the problem, since otherwise it is sufficient to swap the indexing of the intensity functions W_1 (u) and W_2 (u). Note that in all the examples we considered in Section 4, the optimal solutions were not internal switchings. This raises a natural question: is it possible to have such a set of intensity functions for problem (11), namely W_1 (u) and W_2 (u), for which the optimal solution is an internal switching? It turns out that there is, but finding such an example is a non-trivial task. Various combinations of intensity functions from the classes of convex, concave and convex-concave functions (like x^3) did not lead to optimal internal switching.

Let us consider an even more special case of problem (11), namely, when both jobs are the same, i.e. $W_1(u)=W_2(u)=W(u)$ and $A_1=A_2=a$. Is there a problem of this type, all of whose optimal solutions are internal switchings? Problem (11) takes the following form in this case

$$\min_{u_1,u_2,t,T} T \tag{12}$$

$$W\ (u_1)t + W\ (u_2)(T-t) \geq a,$$
$$W\ (A-u_1)t + W\ (A-u_2)(T-t) \geq a,$$
$$0 \leq t \leq T,$$
$$0 \leq u_1 \leq u_2 \leq A.$$

To find the example that interests us, we give a geometric interpretation of problem (12). Consider a phase space in which the x coordinate is the value W(u), and the y coordinate is the value W(A-u). The curve of this space that corresponds to the values of the parameter u from 0 to A is called the "intensity curve". The coordinates of the points of this curve represent the set of all possible instantaneous distributions of intensities between jobs. This curve is symmetric with respect to the bisector of the positive quadrant of the coordinate plane. We introduce the following notation:

$$k_1 = t, k_2 = T-t, x_1 = W(u_1), y_1 = W(A-u_1), x_2 = W(u_2), y_2 = W(A-u_2).$$

Then problem (12) takes the form:

$$\min_{u_1,u_2,k_1,k_2} (k_1+k_2) \tag{13}$$

$$k_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \geq \begin{pmatrix} a \\ a \end{pmatrix}$$
$$0 \leq k_1, 0 \leq k_2, 0 \leq u_1 \leq u_2 \leq A.$$

Representation (13) makes it possible to geometrically “see” the solution to the problem. We have a parametrically specified curve in the x and y axes, namely: x=W(u), y=W(A-u), $0 \leq u \leq A$. Further we will call it the “intensity curve”. An example of such a curve is shown in Fig. 1. What properties does this curve have?

1) It always starts at the point (0, W(A)), which corresponds to the parameter value u=0, and ends at the point (W(A), 0), which corresponds to the parameter value u=A.

2) The curve is symmetrical with respect to the bisector of the first quadrant of the coordinate plane (y=x). This follows from the fact that the points corresponding to the parameter u and A-u, namely, (W(u), W(A-u)) and (W(A-u), W(u)) have the indicated symmetry.

3) The function y(x) is decreasing. This follows from the fact that if $u_1<u_2$, then $W(A-u_1)>W(A-u_2)$.

As we see, the intensity curve in Fig. 1 has the above properties.

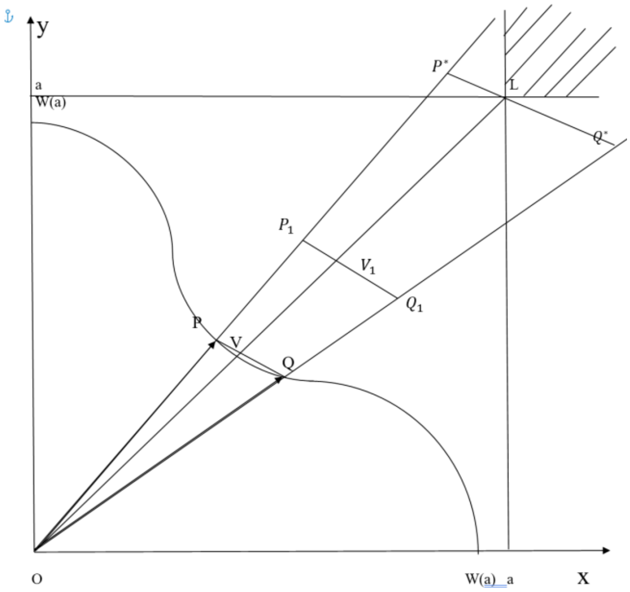

Fig. 1. Geometric representation of problem (13) (case 1).

According to the statement (13), we have 4 “control levers” at our disposal: variables u_1, u_2, which determine the vectors (x_1, y_1) and (x_2, y_2), and variables k_1, k_2. Let us consider the isolines of the functional for fixed values of the parameter u_1, u_2, which correspond to points P and Q in Fig. 1. The isolines are segments between the rays [O, P) and [O, Q) parallel to the segment [P, Q], for example $[P_1, Q_1]$ (Fig. 1). Let us consider the case when P and Q are on different sides of the bisector (case 1). The value of the functional for the segment [P, Q] is 1, and for the segment $[P_1, Q_1]$ the value is $|OV_1|/|OV|$. As is easy to see, the fulfillment of the conditions of problem (13) means that the segment parallel to the segment [P, Q] has a non-empty intersection with the shaded rectangular region with the vertex at point L (Fig. 1). Such a segment, which corresponds to the minimum value of the functional, as can be seen from Fig. 1, is the segment $[P^*, Q^*]$, and the corresponding value of the functional is equal to $|OL|/|OV|$. From this, in turn, it follows that the minimum value of the functional is achieved at the maximum value of the length $|OV|$.

Thus, the geometric solution to the problem is to find on the intensity curve such points P and Q for which the point V will be maximally distant from the point O. Note also that in our consideration, points P and Q were chosen to lie on different sides of the bisector of the positive quadrant (case 2). Let us further study the question of the advisability of choosing points P and Q on the same side of the bisector (Fig. 2).

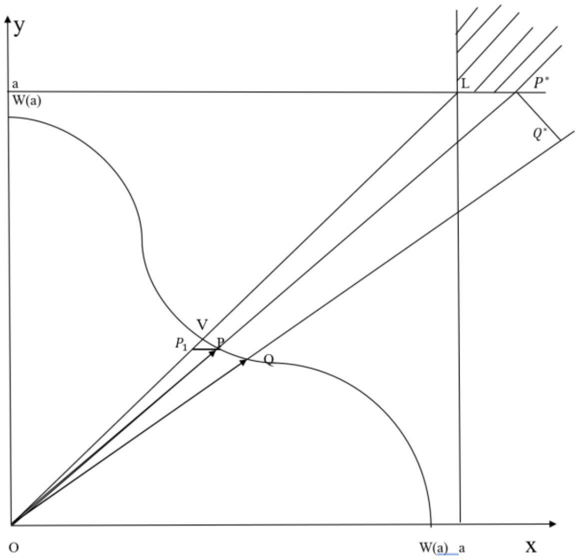

Fig. 2. Geometric representation of problem (13) (case 2).

In this case, the position of the isoline of the functional, at which the minimum of the functional is achieved, is the segment $[P^*, Q^*]$, parallel to the segment [P, Q] (Fig. 3), and the value of the functional in this case is equal to $|OP^*|/|OP|$. Next, we construct the segment $[P, P_1]$ parallel to the segment $[P^*, L]$. Now we compare the current value of the minimum functional for this choice of points P and Q with the result corresponding to the choice of P=Q=V, where V is the intersection point of the intensity curve with the bisector of the positive quadrant. For given choice of points, the minimum value of the functional is $|OL|/|OV|$. The following chain of inequalities and their consequences is valid:

$$|OP_1| < |OV| \rightarrow \frac{|OL|}{|OV|} < \frac{|OL|}{|OP_1|} = \frac{|OP^*|}{|OP|}.$$

This means that the choice of P=Q=V is always better than the choice of arbitrary points P and Q lying on the same side of the bisector of the positive quadrant. Thus, when searching for a geometric solution, it is sufficient to consider only the points P and Q on different sides of the bisector, and in the limiting case, coinciding and lying on it. Next, we study several special cases.

1) Let the function W(u) be convex. Then, using the differentiation formula for a parametrically defined function [15], we have

$$\frac{d^2y}{dx^2} = \frac{\ddot{W}(A-u)\dot{W}(u) + \ddot{W}(u)\dot{W}(A-u)}{\dot{W}(u)^3} \quad (14)$$

Since $\dot{W}(u) \geq 0$, since the function W(u) is non-decreasing and $\ddot{W}(u) \geq 0$, since the function W(u) is convex, we have

$$\frac{d^2y}{dx^2} \geq 0,$$

and therefore, the intensity curve is convex (Fig. 6).

In this case, applying the above geometric ideas, it is obvious that the optimal position of the points P^* and Q^* should be chosen as shown in Fig. 3. In this case, $u_1^* = 0$, and $u_2^* = A$. This corresponds to the sequential execution of work using the full resource. Let us demonstrate this further using a specific numerical example. An example for this case can be example 1.1, considered earlier, with the difference that the intensity functions in it are convex, but different from each other. Its numerical solution, as we have seen, really suggests sequential execution of work as an optimal solution. In addition, let us construct an intensity curve for example 1.1. (Fig. 4). As we see, the symmetry property is violated, but the convexity property of the intensity curve graph is preserved. And therefore, our geometric reasoning remains valid.

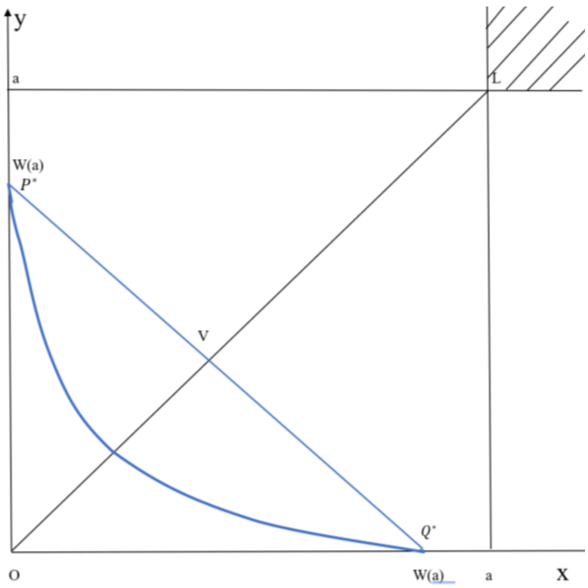

Fig. 3. The case of a convex intensity curve.

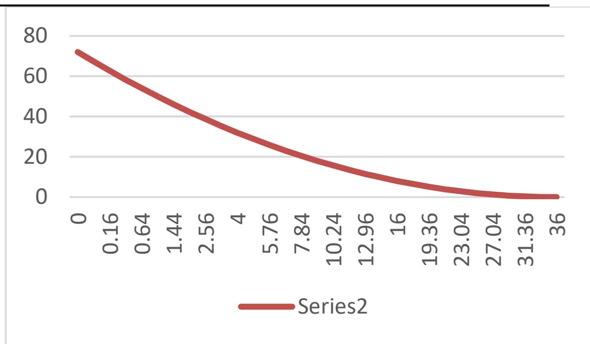

Fig. 4. Intensity curve of example 1.1.

2) Let the function W(u) be concave. Again, we use formula (14). Since $\dot{W}(u) \geq 0$, since the function W(u) is non-decreasing $\ddot{W}(u) \leq 0$, since the function W(u) is concave, we have

$$\frac{d^2y}{dx^2} \leq 0,$$

and therefore, the intensity curve is concave (Fig. 5).

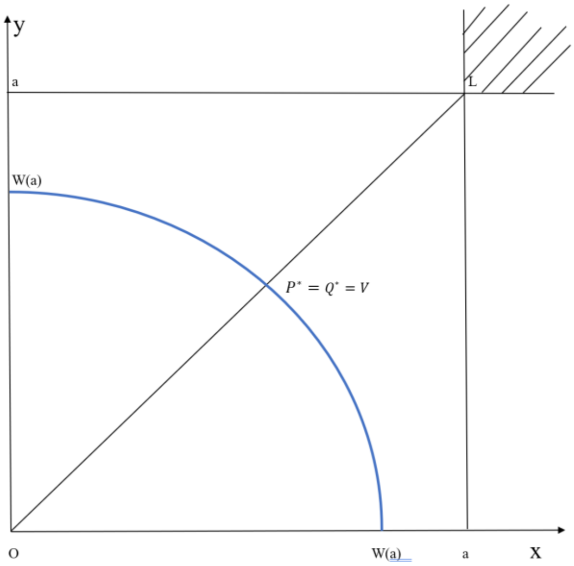

Fig. 5. The case of a concave intensity curve.

In this case, applying the above geometric ideas, it is obvious that the optimal position of the points P^* and Q^* should be chosen as shown in Fig. 2. In this case, $u_1^* = u_2^* = u^* = 0{,}5\,A$ A. In the case where the intensity functions are concave but not identical, u^* is the root of the equation $W_1(u)=W_2(A\text{-}u)$, which, due to the properties of the intensity functions, has a unique solution. This corresponds to the parallel execution of work with the distribution of the resource u^* and $A - u^*$. We will demonstrate this further on a specific numerical example. An example for this case can be example 1.2, considered earlier, with the difference that the intensity functions in it are concave, but different from each other. Its numerical solution, as we have seen, is indeed the optimal solution is offered by parallel execution of works with synchronous completion. In addition, we will construct the intensity curve for example 1.2. (Fig. 6). As we can see, the symmetry property is violated, but the property of concavity of the intensity curve graph is preserved. And therefore, our geometric reasoning remains valid.

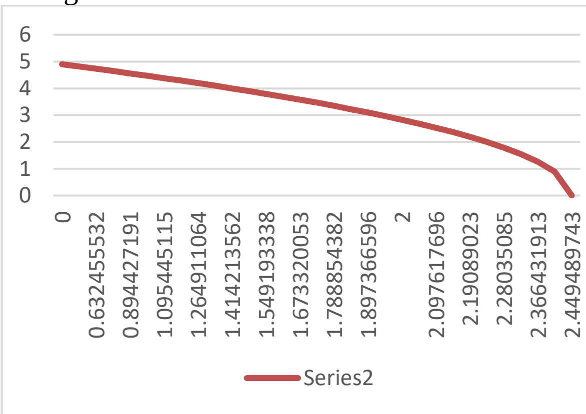

Fig. 6. Intensity curve for example 1.2.

Let us now return to the question posed earlier: is it possible to have such a set of intensity functions for problem (11), namely $W_1(u)$ and $W_2(u)$, for which the optimal solution is an internal switching? From the consideration of cases 1) and 2) it is clear that in the class of convex or concave intensity functions this is impossible. From the geometric approach we have developed above, it is clear that internal switching is possible for a convex-concave intensity curve, as shown, for example, in Fig. 7. The points P^* and Q^*, constructed on the basis of the geometric approach, are also shown there. To verify the reality of such a case, let us consider a specific numerical example 5.1.

Example 5.1

$$W_1(u) = W_2(u) = u + \frac{1}{6}\sin(\pi u),$$
$$A = 3,\ A_1 = 20, A_2 = 20.$$

The intensity functions are convex-concave. The intensity curve is shown in Fig. 8 and is also a convex-concave function. The results of the solution are as follows t=6, T=12, u_1=2,5, u_2=0,5. As can be seen from the obtained result, the solution to this problem the solution to this problem is a complete internal switching, which is what we expected based on the geometry of the intensity curve.

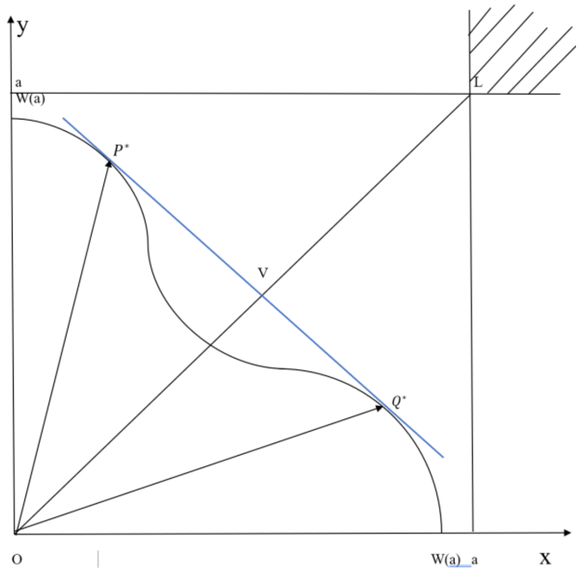

Fig. 7. Convex-concave intensity curve.

Fig. 8. Intensity curve of example 5.1.

Conclusion

The problem of optimal allocation of reentrant resources for performing a set of independent jobs was considered. For the simplest case, namely, the case of homogeneous linear productivity functions, an analytical solution was obtained and substantiated. For the case of nonlinear homogeneous productivity functions, some resource allocation algorithms were studied and their non-optimality was shown in the general case. The question of the existence of full internal switching as a solution to the problem in the case of two jobs was posed. The concept of an intensity curve was introduced and a geometric optimality criterion was constructed on its basis. A specific numerical example with full internal switching as an optimal solution was given.

References

1. DAVYDOV E.G. Games, graphs, resources. - M.: Radio and communication, 1981. - 113 p.

2. BURKOV V.N., GORGIDZE I.A., LOVETSKY S.E. Applied problems of graph theory. - Tbilisi: Metsniireba, 1974. - 234 p.

3. KOSORUKOV O.A., LEMTYUZHNIKOVA D.V., MISHCHENKO A.V. Methods and models of project resource management under uncertainty // Bulletin of the Russian Academy of Sciences. Theory and control systems. - 2023. - No. 3. - P. 38-56.

4. BORISOV I.A., KOSORUKOV O.A., MISHCHENKO A.V., TSURKOV V.I. Optimization of Enterprise Production Programs Taking Into Account Uncertainty // Journal of Computer and Systems Sciences International, 2024, Vol. 63, No. 4, pp. 663–678.

5. S.A. BARKALOV, S.I. MOISEEV, A.I. SUKHOMLINOV Methodology for optimizing a network schedule by redistributing resources. Project management in construction. 2021. No. 3 (24). P. 5-30.

6. KOSORUKOV O.A., LEMTYUZHNIKOVA D.V. An Optimal Allocation Algorithm for Reentrant Resources on Network Graphs // CONTROL SCIENCES Scientific Technical Journal, 2024, Vol. 2, pp. 2-8.

7. LYAKHOV O. A. Resources in network planning of complex work sets // Problemy informatiki. – 2013. – No. 1 (18). – P. 27–36.

8. RAZUMIKHIN B. S. The Problem on Optimal Resources Distribution // Automation and Remote Control. – 1965. – Vol. 26, no, 7. – P. 1227–1246.

9. V. N. BURKOV, L. V. ROSSIKHINA, A. P. VYUNOV, L. V. ROGOVAYA The problem of optimal distribution of teams of specialists. Automation and Telemechanics, 2019. No. 1. P. 116-125.

10. MISHENKO, A., KOSORUKOV, O., SVIRIDOVA, O. Optimization of Works Management of the Investment Project // The 2nd International & European Conference “Modeling and Simulation of Social-Behavioural Phenomena in Creative Societies” (MSBC-2022). – Vilnius, 2022. – P. 201–217.

11. GEHRING, M., VOLK, R., SCHULTMANN, F. Discrete Optimization on the Integration of Diverging Material Flows into Resource-Constrained Project Scheduling // European Journal of Operational Research. – 2022. – Vol. 303, iss. 3. – P. 1071–1087.

12. BIANCO, L., CARAMIA, M., GIORDANI, S. Project Scheduling with Generalized Precedence Relations: A New Method to Analyze Criticalities and Flexibilities // European Journal of Operational Research. – 2022. – Vol. 298, iss. 2. – P. 451–462.

13. LIMA, V. L., ALVES, C., CLOUTIAUX, F. Arc Flow Formulations Based on Dynamic Programming: Theoretical Foundations and Applications // European Journal of Operational Research. – 2022. – Vol. 296, iss. 1. – P. 3–21.

14. KOSORUKOV O. A., BELOV A. G. The problem of resource management on network graphs as an optimal control problem. – Vestn. Mosk. Univ. Ser. 15, Vychislit. matematika i cybernetics. – 2014. - No. 2, - P. 29-33.

15. MINORSKY V. P. Collection of problems in higher mathematics. Moscow, Nauka 1969, -352 p.